Shortest Paths

Input: Directed graph $G = (V, E)$,

each edge (u,v) in E has a weight $wt(u,v) > 0$,

fixed vertex s (source).

Output: for each vertex v:

$\delta(s,v)$ = length of a shortest path from s to v.

Approach: for each vertex v, maintain variable

$d(v)$ = length of a shortest path from s to v found so far.

Start: $d(s) = 0$, for every vertex $v \neq s$: $d(v) = \infty$.

Loop: Pick a vertex u for which $d(u) = \delta(s,u)$.

For each edge (u,v):

Shortest paths in a directed graph $G = (V, E)$.

each edge (u,v) in E has a weight $wt(u,v) > 0$.

source vertex s.

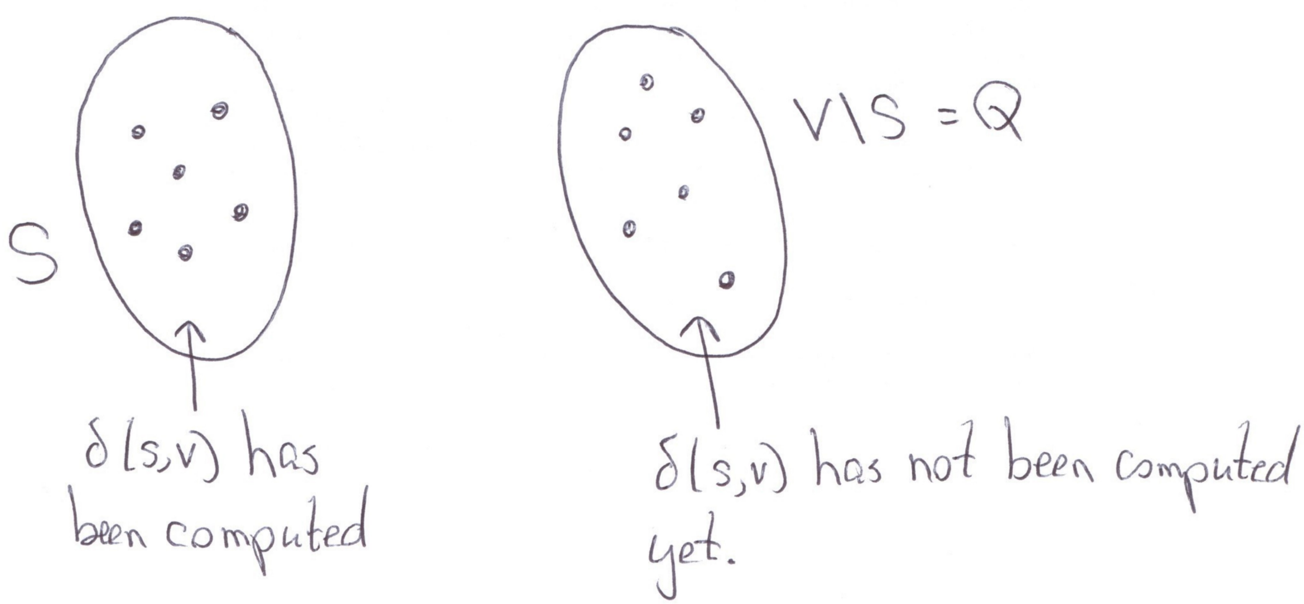
<u>Goal</u>: $\delta(s,v)$ for each vertex v.

<u>Approach</u>: ① for each vertex v, maintain a variable

$d(v)$ = length of a shortest path from s to v found so far.

② maintain $S \subseteq V$ such that for all $v \in S$:

$d(v) = \delta(s,v)$ (i.e., we know $\delta(s,v)$)



Start: $S = \phi$, $Q = V$,

$d(s) = 0$,

$d(v) = \infty$ for each vertex $v \neq s$.

One iteration: grow S, by moving one vertex u from Q to S.

Which vertex u do we move:

u = vertex of Q for which $d(u)$ is minimum.

Later, we prove: for this vertex u: $d(u) = \delta(s,u)$.

for each edge (u,v):

$d(v) = \min(d(v), d(u) + wt(u,v))$.

Dijkstra (1959)

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for each v ∈ V : d(v) = ∞;
d(s) = 0; S = φ; Q = V;
while Q ≠ φ :
    u = vertex in Q for which d(u) is minimum;   (*)
    delete u from Q;
    insert u into S;
    for each edge (u,v):
        if d(u) + wt(u,v) < d(v):
            d(v) = d(u) + wt(u,v)
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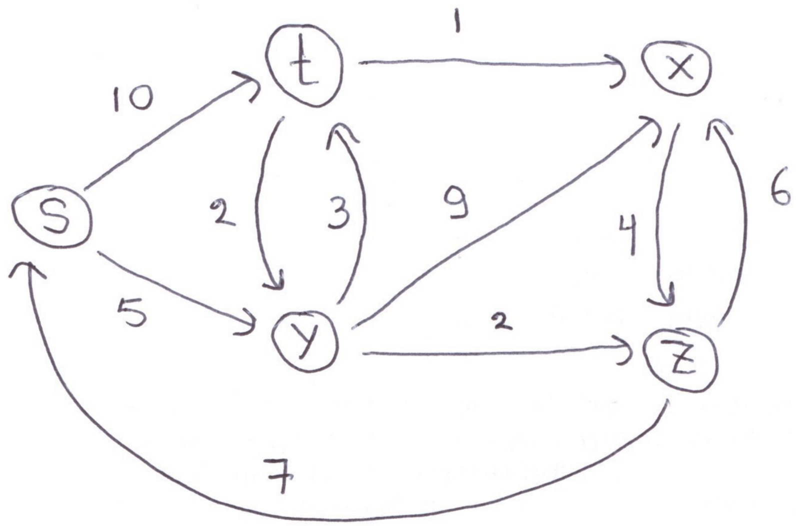
(*) We will prove later that $d(u) = \delta(s,u)$



Q	s	t	x	y	z
d	0	∞	∞	∞	∞

$u = s$

$\delta(s,s) = d(s) = 0$

delete s from Q

update $d(t)$ and $d(y)$

Q	t	x	y	z
d	10	∞	5	∞

$u = y$, $\delta(s,y) = d(y) = 5$

delete y from Q

update $d(t)$, $d(x)$, $d(z)$

Q	t	x	z
d	8	14	7

$u = z$

$\delta(s,z) = d(z) = 7$

delete z from Q

update $d(x)$ and $d(s)$

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Q	t	x
d	8	13

$u = t$

$\delta(s,t) = d(t) = 8$

delete t from Q

update $d(x)$ and $d(y)$

Q	x
d	9

$u = x$

$\delta(s,x) = d(x) = 9$

delete x from Q

update $d(z)$

$Q = \phi$; done.

<u>Running time</u>: $n = |V|$, $m = |E|$

Store Q in a min-heap, where the key of each vertex v is $d(v)$.

<u>Initialization</u>: $O(n)$ (this includes the time to build a heap storing $Q = V$)

One iteration:

* find u and delete it from Q:

  extract_min : $O(\log n)$ time

* for each edge (u,v): update $d(v)$:

  decrease_key : $O(\log n)$ time

Total time for one iteration:

$$O(\log n) + O(\text{outdegree}(u) \cdot \log n)$$

Total running time:

$$O(n) + O\left(\sum_{u \in V} \left(\log n + \text{outdegree}(u) \cdot \log n\right)\right)$$

$$= O(n \log n + m \log n)$$

$$= O((n+m) \log n).$$

CORRECT? * Recall shortest paths have optimal substructure.

<u>Lemma 1:</u> When vertex p is added to S, $d(p) = \delta(s,p)$.

Proof: By induction on the number of vertices in S.

<u>Base case:</u> $S = \{s\}$. Since $d(s) = \emptyset$, and there are no negative edge weights, $d(s) = \delta(s,s)$

<u>Inductive Hypothesis:</u> Let $p = Q_{min}$, that is, for all vertices still in Q, $d(p)$ has the minimum value. Then for all vertices $x \in S$, we assume $d(x) = \delta(s,x)$.

Look at $\delta(s,p)$:

$$s \to a \to b \to c \to d \to p$$

If $a, b, c, d \in S$, then by the inductive hypothesis, $d(d) = \delta(s,d)$. When we added d to S, we would have updated $d(p)$, thus

$$\begin{aligned} d(p) &\leq d(d) + wt(d,p) \\ &\leq \delta(s,d) + wt(d,p) \\ &= \delta(s,p) \quad \text{- because of optimal substructure.} \end{aligned}$$

Else at least one of a, b, c, d are not in S.
without loss of generality, assume b is first vertex on $\delta(s,p)$ that is not in S.

That means $a \in S$, and by our inductive hypothesis, $d(a) = \delta(s,a)$. Because of optimal substructure of shortest paths, $s \to a \to b$ is a shortest path to b. When a was added to S, we updated $d(b)$, so we know

$$\begin{aligned} d(b) &\leq d(a) + wt(a,b) \\ &\leq \delta(s,a) + wt(a,b) \\ &= \delta(s,b) \end{aligned}$$

However, since $b \notin S$, it must be that $b \in Q$.
Since $p = Qmin$, we have

$$\text{①} \quad \begin{aligned} d(p) &\leq d(b) \\ &\leq \delta(s,b). \end{aligned}$$

The shortest path from b to p must be of length $\emptyset$ (since no negative edge weights). i.e. $\delta(b,p) = \emptyset$.

$$\begin{aligned} \delta(s,p) &= \delta(s,b) + \delta(b,p) \\ &= \delta(s,b) \\ &= d(p) \quad (\text{from ①}) \end{aligned}$$