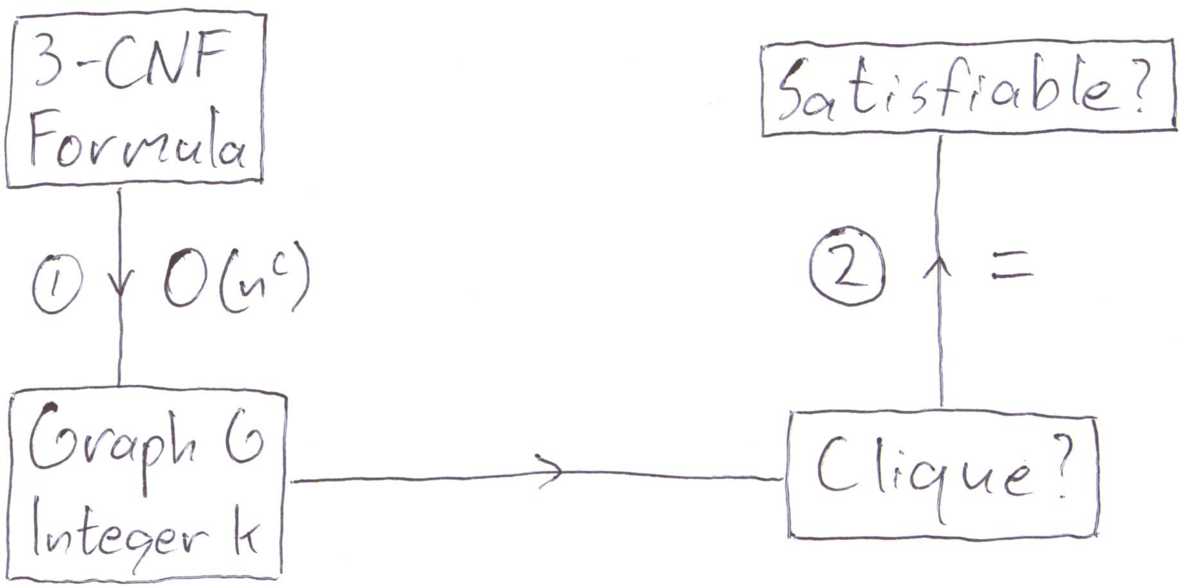
Claim: Clique is NP-complete.

* Clique is in NP:
  Certificate = Set of vertices $V' = \{v_1, \dots, v_k\}$.
  Verification:
  - check that $|V'| = k$
  - check that there is an edge between any two vertices in V'.

  This can be done in $O(n^2)$ time. ✓

* Clique is NP-hard: by reduction from 3-SAT.
  We need to show 3-SAT $\leq_p$ Clique.



Given a 3-CNF formula φ, we need to find a graph G and an integer k, such that:

① We can compute G and k in time polynomial in the size of φ.

② G has a clique of size $k \iff \varphi$ is satisfiable

To build G:

- add 3 vertices for each clause (one for each literal)
- add an edge between each pair of vertices in <u>different</u> clause groups, <u>unless</u> they represent an $(x, \neg x)$ pair.

Let k = #clauses in φ.

①: There are $\leq 3k$ vertices in G and the edges can be computed in $O(k^2)$ time.

② $\Rightarrow$: Suppose G has a clique V' of size k.
Since vertices in the same clause group aren't connected to each other, V' must contain exactly one vertex from each clause group. Set the literals corresponding to vertices in V' to true (so x_i should be false if $\neg x_i \in V'$). No $(x, \neg x)$ pair is connected by an edge, so this is a valid truth assignment that makes every clause true.
$\Leftarrow$: Suppose φ is satisfiable. Then at least one literal of every clause is true, for some assignment. For each clause, pick a vertex corresponding to a true literal. Since the assignment cannot make both x and $\neg x$ true, these vertices form a clique of size k. □

0/1-ILP (Integer Linear Programming)

Input: - A set of variables $\{x_1, \dots, x_n\}$

- A set of linear inequalities in these variables, with integer coefficients. $(A\bar{x} \geq \bar{b})$

~~Output:~~

Question: Does there exist some assignment of 0/1 to the variables such that all inequalities hold?

Example:

$$x_1 + x_2 \geq 1 \qquad\qquad x_1 = 1 \text{ and } x_2 = 0$$
$$x_1 - 2x_2 \geq 0 \qquad\qquad \text{satisfies all inequalities}$$

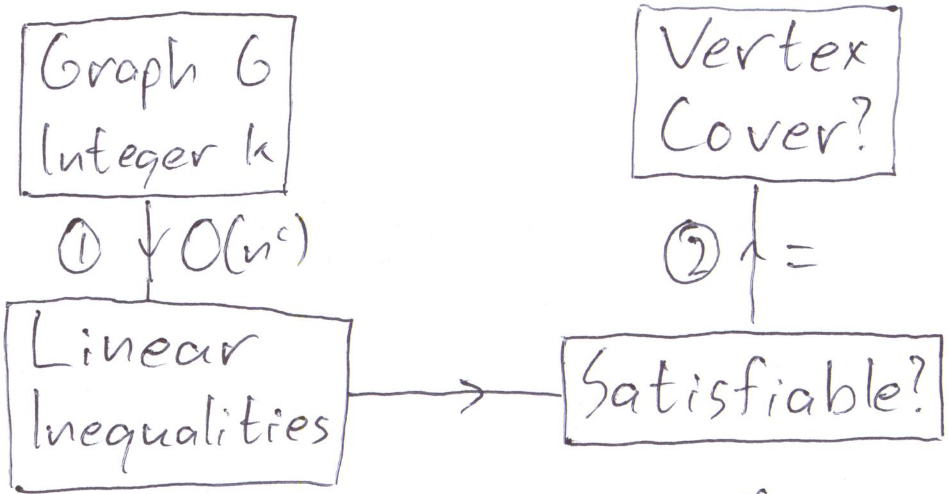
Claim: 0/1-ILP is NP-complete.

* 0/1-ILP is in NP:
  Certificate = assignment of 0/1 to vars.
  Verification:
  - check each inequality

  This can be done in $O(nm)$ time, where m = #inequalities.
* 0/1-ILP is NP-hard:

  We show that Vertex Cover $\leq_p$ 0/1-ILP.

  (We showed Clique $\leq_p$ Vertex Cover in Lecture 19, so VC is NP-hard.)



① Introduce a variable x_v for each vertex. $v \in V$
For each edge (u,v), add the constraint

$$x_u + x_v \geq 1$$

Finally, add the constraints

(*) $\sum_{v \in V} x_v \geq k$ and (**) $\sum_{v \in V} -x_v \geq -k$ $\left(\} \Rightarrow \sum_{v \in V} x_v = k \right)$

This can be computed in $O(|V|+|E|)$ time.

② We need to show that this program is satisfiable if and only if G has a vertex cover of size k.

$\Rightarrow$: Suppose the program is satisfiable. Let $V' = \{v \in V \mid x_v = 1\}$. Because of the edge constraints, at least one endpoint of each edge is in V', so V' is a vertex cover of size k (due to (*) and (**)).

$\Leftarrow$: Suppose G has a vertex cover V' of size k. Set x_v to 1 iff $v \in V'$.
Since V' is a vertex cover, all edge constraints are satisfied. The other two constraints are satisfied because $|V'| = k$, so the program is satisfiable. $\square$

<u>Integer Linear Programming</u>

<u>Input</u>: A set of n variables and m linear inequalities in these variables, with integer coefficients.

<u>Question</u>: Does there exist an assignment of integers to the variables such that all inequalities hold?

<u>Claim</u>: 0/1-ILP $\leq_p$ ILP

<u>Proof Sketch</u>: Copy the program, but add the constraints

$$x \geq 0 \quad \text{and} \quad -x \geq -1 \quad (0 \leq x \leq 1)$$

for each variable.