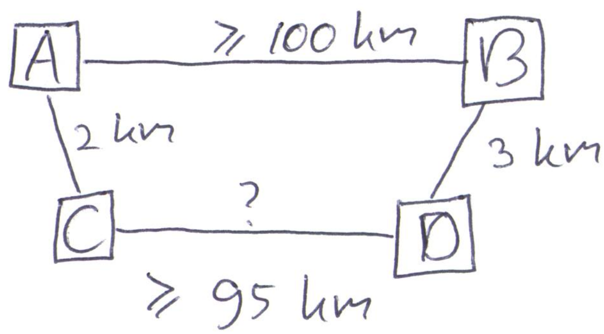
# Reductions

If we want to prove that $P \neq NP$, we have to show that there exists a problem $X$ in NP that is not in P.

This $X$ must be "very difficult".
$\Rightarrow$ Look at the "most difficult" problems in NP.

How do we compare the difficulty of problems?
$\Rightarrow$ Reductions

Analogy: ~~Suppose you~~

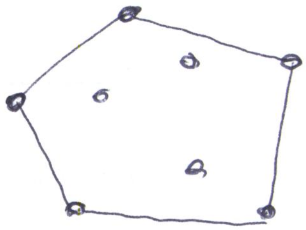



This works for problems as well.

## Convex Hull

Input: Set $P$ of $n$ points in the plane.
Output: Vertices of the smallest convex polygon containing $P$, in clockwise order.

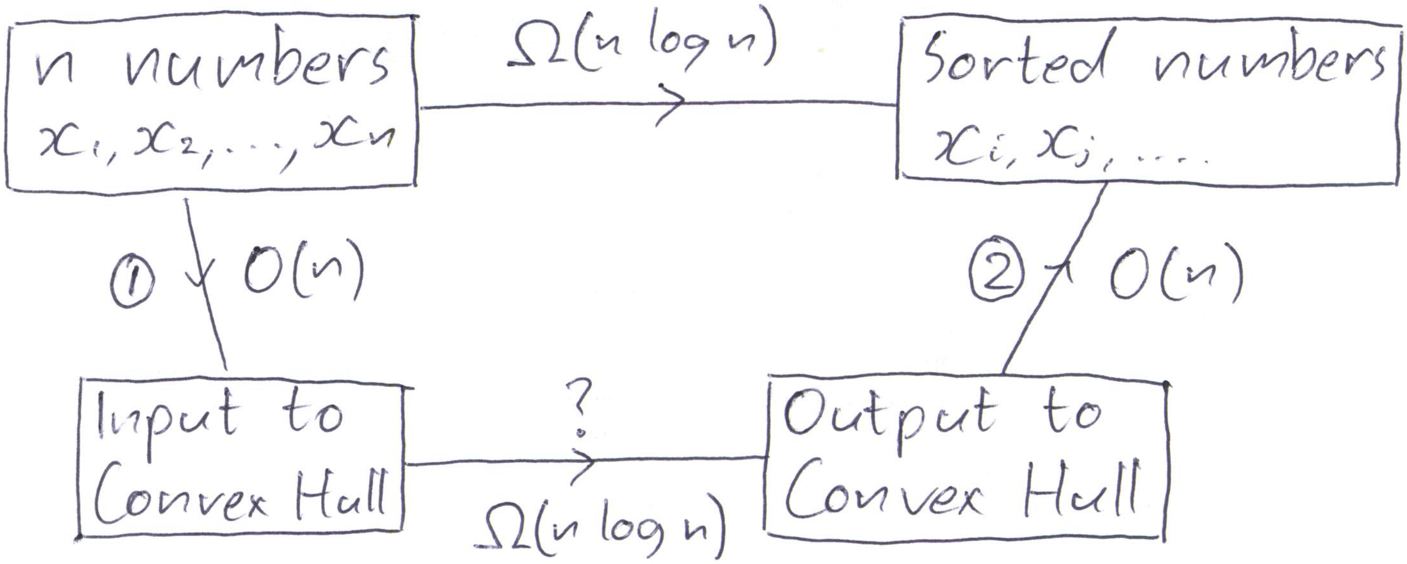
**Example:**



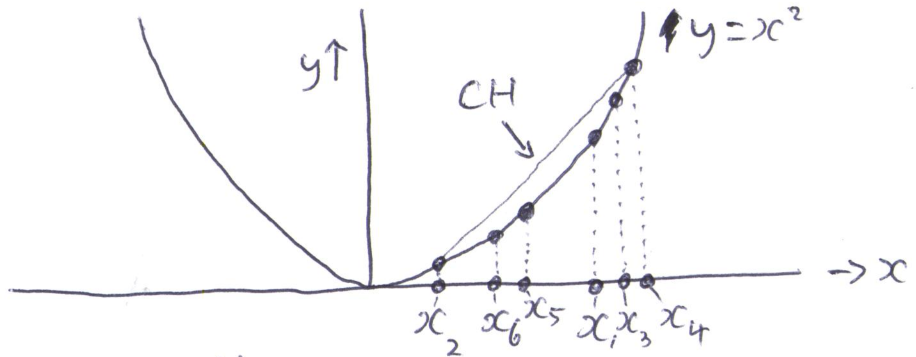
(Imagine snapping an elastic band around a bunch of nails.)

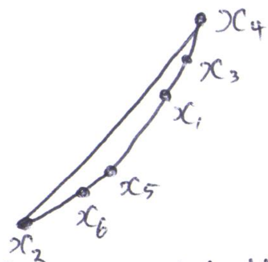
**Claim:** Solving Convex Hull requires $\Omega(n \log n)$ time

**Proof:** Recall that sorting $n$ numbers takes $\Omega(n \log n)$ time in the worst case.



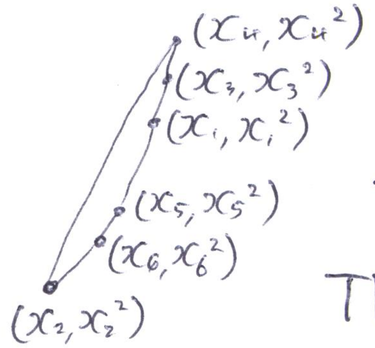

① $x_1, x_2, \ldots, x_n \rightarrow (x_1, x_1^2), (x_2, x_2^2), \ldots, (x_n, x_n^2)$




Convex Hull

② 




$\rightarrow x_2, x_6, x_5, x_1, x_3, x_4$

Thus, Convex Hull takes $\Omega(n \log n)$ time. $\square$

We showed that computing the convex hull is at least as hard as sorting.

We can use the same technique for problems in NP:

$$\begin{tikzcd}
\boxed{\text{Input to } A} \arrow[r] \arrow[d, "\text{①}"', "O(n^c)"] & \boxed{\text{Answer for } A} \\
\boxed{\text{Input to } B} \arrow[r] & \boxed{\text{Answer for } B} \arrow[u, "\text{②}", "="']
\end{tikzcd}$$

① In polynomial time, we ~~can~~ transform an input $I$ for $A$ into an input $I'$ for $B$.

② We prove that the answer for $I'$ is YES if and only if the answer for $I$ is YES.

This is a reduction <u>from</u> A <u>to</u> B.
We write this as $A \leq_p B$.
You can read this as "B is at least as difficult as A."

**Example:**

**Independent Set**

**Input:** Graph $G$ and integer $k$.

**Question:** Does $G$ have an independent set of size $k$?

no two vertices connected by an edge

**Clique**

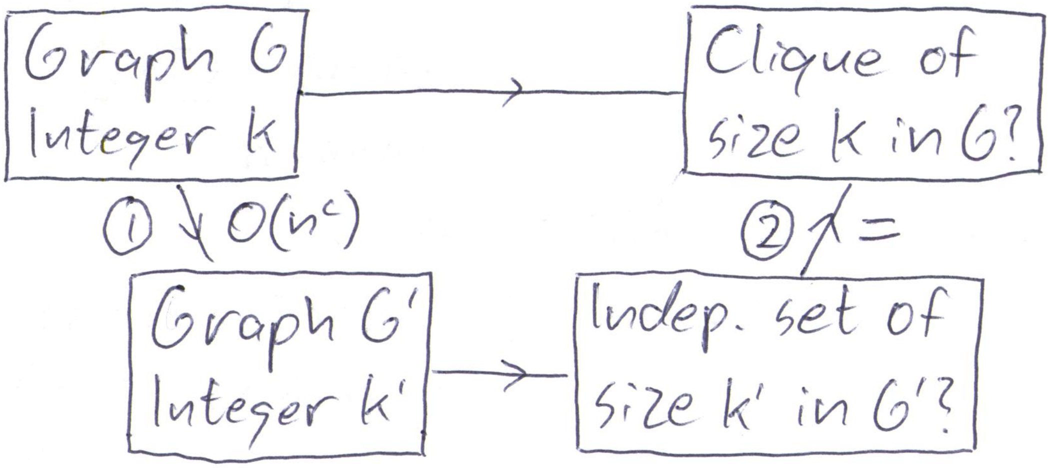
**Input:** Graph $G$ and integer $k$.

**Question:** Does $G$ have a clique of size $k$?

all pairs of vertices connected by an edge

**Claim:** Clique $\leq_p$ Independent Set

**Proof:**




We need a graph $G'$ and integer $k'$ s.t.:

① We can build $G'$ in polynomial time

② $G'$ has an independent set of size $k'$

$\Longleftrightarrow$

$G$ has a clique of size $k$

Take

$G'$ = complement of $G$ (edge $(x,y)$ in $G'$ iff no edge $(x,y)$ in $G$)

$k' = k$.

① Inverting the presence of every edge takes $O(|V|^2)$ time.

② $\Rightarrow$: If $G'$ has an independent set of size $k$, those same vertices form a clique in $G$, since edges not present in $G'$ are in $G$.

$\Leftarrow$: If $G$ has a clique of size $k$, those same vertices form an indep. set in $G'$. □

**Example:** Vertex Cover

**Input:** Graph $G$ and integer $k$.

**Question:** Does $G$ contain a vertex cover of size $k$?

subset $V' \subseteq V$ such that every edge has at least 1 endpoint in $V'$

**Claim:** Clique $\leq_L$ Vertex Cover

**Proof:** We need a graph $G'$ and integer $k'$, s.t.:

① We can build $G'$ in polynomial time

② $G'$ has a vertex cover of size $k'$

$$\Longleftrightarrow$$

$G$ has a clique of size $k$