Shortest Paths in DAGs

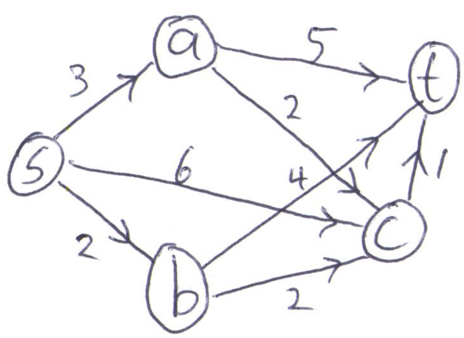
<u>Input</u>: - Directed acyclic graph G with <u>edge weights</u>

- Vertices s and t

edge (u,v) has weight $w(u,v)$.

<u>Output</u>: Total weight of the shortest (minimum weight) path from s to t in G.

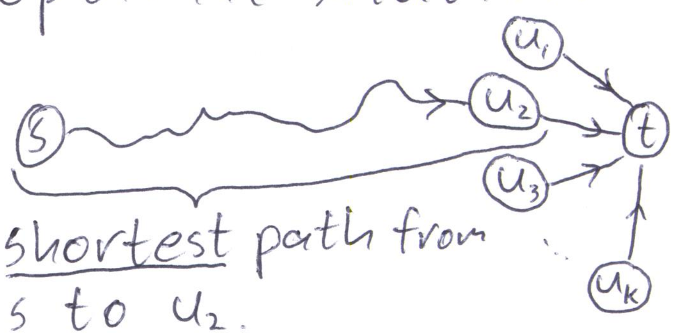
<u>Example</u>:



$s \to b \to c \to t$ is shortest, with total weight 5.

There could be $\Omega(2^{|V|})$ paths from s to t, so we can't just try all of them. Let's try DP.

- ~~Optimal substructure?~~ Structure of the optimal solution:



Let $u_1, \ldots, u_k$ be all vertices with an edge to t.

The last edge on the shortest path is (u_i, t), for some index i.

$\Rightarrow$ The shortest path from s to t contains the shortest path from s to some u_i.

- Recurrence:

  Let $d(v)$ be the length of the shortest path from s to v.

$$d(v) = \begin{cases} 0 & \text{if } v = s \\ \underbrace{\min_{(u,v)\in E}}_{\substack{\text{edges}\\ \text{to } v}} \{d(u) + w(u,v)\} & \text{if } v \neq s \end{cases}$$

- Solve recurrence bottom-up:

  We want to compute $d(v)$ for each vertex v, in such an order that we already computed all $d(u)$ for vertices u with an edge to v.

  This is exactly what topologicalsort does!

<u>Algorithm</u> ShortestPath(G, s, t):

$V \leftarrow$ TopoSort(G)	$O(\lvert V\rvert + \lvert E\rvert)$
$d[s] \leftarrow 0$	$O(1)$
for $i \leftarrow 1$ to n do	$\cancel{O(\lvert V\rvert)}$ $\lvert V\rvert \times O(1)$
if $V[i] \neq s$ then	$\lvert V\rvert \times O(1)$
$d[V[i]] \leftarrow \infty$	$\lvert V\rvert \times O(1)$
for each edge $(u, V[i])$ do	$\lvert E\rvert \times O(1)$
$d[V[i]] \leftarrow \min(d[V[i]], d[u] + w(u, V[i]))$	$\lvert E\rvert \times O(1)$
return $d[t]$	$O(1)$
	+
	$O(\lvert V\rvert + \lvert E\rvert)$

- Efficiency?

  $O(|V|+|E|)$, assuming we can quickly find all incoming edges. (How?)

- Example

Longest Common Subsequence

~~Seq~~ Input: Sequences X and Y.

Output: Length of the longest common subsequence of X and Y.

Example: $X=[a,\underline{b},\underline{c},b,\underline{d},a,\underline{b}]$

$Y=[b,d,c,a,b,a]$

- $[b,c,d,b]$ is a subsequence of X, but not of Y.
- $[a,b,a]$ is a subsequence of both: a common subsequence
- The longest common subsequence has length 4: $[b,c,b,a]$, or $[b,d,a,b]$.